ALL SAINTS' COLLEGE

WA Exams Practice Paper B, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ____________________________

Your name ____________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

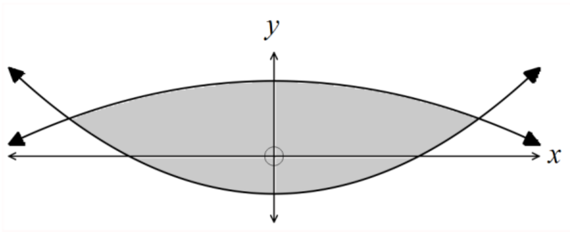
Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(6 marks)**

Two even functions are defined by $f(x) = x^2 - 2$ and $g(x) = 4 - \frac{1}{2}x^2$. The region between them is shaded on the diagram below.



Show that the exact volume of the solid formed by rotating the region enclosed by $f(x)$ and $g(x)$ about the y-axis is 12π cubic units.

$x^2 - 2 = 4 - \frac{1}{2}x^2$

$x = \pm 2 \Rightarrow y = 2$

$f(0) = -2,\ g(0) = 4$

$$V = \pi\int_{-2}^{2}(y+2)\,dy + \pi\int_{2}^{4}(8-2y)\,dy$$

$$= \pi\left(\left[\frac{y^2}{2} + 2y\right]_{-2}^{2} + \left[8y - y^2\right]_{2}^{4}\right)$$

$$= \pi\big((2+4) - (2-4) \quad + \quad (32-16) - (16-4)\big)$$

$$= 12\pi \text{ cubic units}$$

Question 9 **(5 marks)**

Two small objects, A and B, have initial positions of $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $5\mathbf{i}-10\mathbf{j}+6\mathbf{k}$ respectively. They are moving with constant velocities of $\mathbf{v}_A = 2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{v}_B = \mathbf{i}+2\mathbf{j}-\mathbf{k}$.

Show that their paths intersect, stating the point of intersection and whether or not they collide at this point.

$$r_A = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + a\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},\ r_B = \begin{bmatrix} 5 \\ -10 \\ 6 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Equate coeffs for intersection:

$\mathbf{i}$-coeffs : $1+2a = 5+b$

$\mathbf{j}$-coeffs : $2-a = -10+2b$

Solve to get $a = 4,\ b = 4$

Check $\mathbf{k}$-coeffs : $-2+4 = 6-4 \Rightarrow$ Consistent, so intersect.

Collide as times (a and b) the same.

$$r = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ 2 \end{bmatrix}$$

Intersect at (9, -2, 2)

Question 10 **(7 marks)**

An internet service provider plans to sample the volume of content downloaded per day by customers subscribing to their ADSL20 plan. From recent research, the company knew that the standard deviation of the volume of downloads per customer was 1.4 GB.

(a) Determine how large a sample the company should take in order to be 90% confident that the mean volume of downloads per customer calculated from their sample is within 0.25 GB of the true population mean. (2 marks)

$$n = \left(\frac{1.645 \times 1.4}{0.25}\right)^2$$
$$= 84.9$$

Take a sample of at least 85 customers

(b) A random selection of 25 subscribers was made and the total volume downloaded by these customers over a 24 hour period was 120GB. Calculate a 95% confidence interval for the mean volume of content downloaded per day by a customer. (3 marks)

$$\bar{X} = 120 \div 25$$
$$= 4.8$$

$$z\frac{\sigma}{\sqrt{n}} = 1.96 \times 1.4 \div \sqrt{25}$$
$$= 0.549$$

$$4.8 \pm 0.549 = (4.251,\ 5.349)$$

(c) If the company repeated the random sampling process and subsequent 95% confidence interval calculations from part (b) a total of 40 times, how many of the intervals calculated would you expect to contain the true population mean? Explain your answer. (2 marks)

38 of the intervals.

The 95% level of confidence refers to the probability that the interval contains the population mean.

Question 11 **(7 marks)**

As water slowly leaks out of a hole in the bottom of a cylindrical tank, the rate of change of the depth of water in the tank, h cm, can be modelled by the differential equation $\frac{dh}{dt} = -k\sqrt{h}$, $0 \le t \le 900$ where t is the time in seconds and $k = 0.003$.

Initially, the depth of water in the can was 36 cm.

(a) Show that $h = \left(6 - \frac{kt}{2}\right)^2$. (4 marks)

$$\int \frac{dh}{\sqrt{h}} = \int -k dt$$
$$2\sqrt{h} = -kt + c$$
$$2\sqrt{36} = c \;\Rightarrow\; c = 12$$
$$\sqrt{h} = \frac{12 - kt}{2}$$
$$h = \left(6 - \frac{kt}{2}\right)^2$$

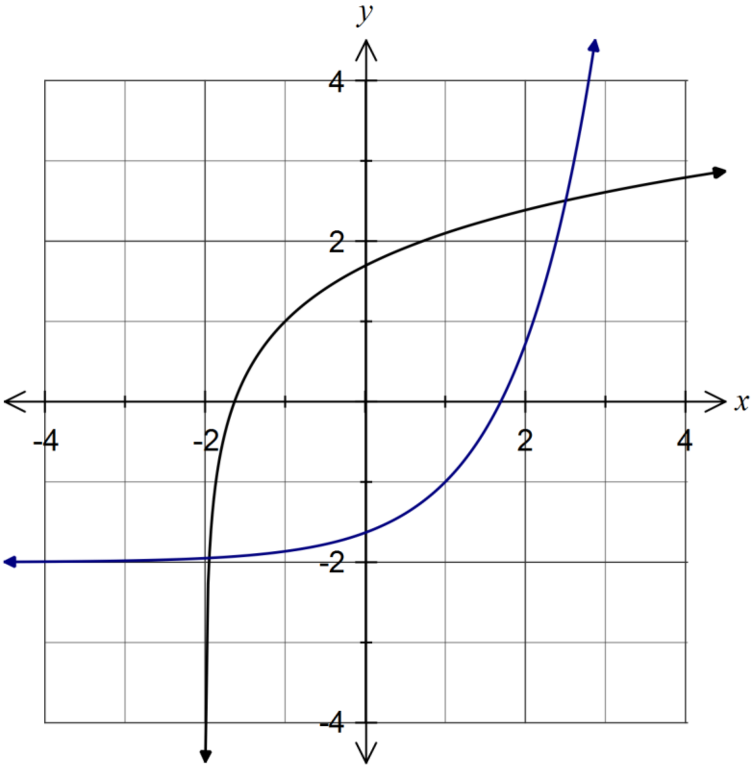
(b) Determine the rate at which the depth of water in the tank is decreasing after ten minutes. (3 marks)

$$h(600) = \left(6 - \frac{0.003 \times 600}{2}\right)^2$$
$$= 26.01$$
$$\frac{dh}{dt} = -0.003 \times \sqrt{26.01}$$
$$= -0.0153$$

Decreasing at 0.0153 cm/s

Question 12 **(7 marks)**

The graph of the function f , defined by $f(x)=\ln(x+2)+1$, is shown below.



(a) Determine the function f^{-1}, the inverse of f , and add a sketch of the graph of $y=f^{-1}(x)$ to the axes above. (4 marks)

$$x=\ln(y+2)+1$$
$$e^{x-1}=y+2$$
$$f^{-1}(x)=e^{x-1}-2$$

(b) Determine the area of the region bounded by the graphs of f and f^{-1}. (3 marks)

Functions f and f^{-1} intersect when $x=-1.94753$ and $x=2.50524$.

$$\text{Area}=\int_{-1.94753}^{2.50524}\left((\ln(x+2)+1)-(e^{x-1}-2)\right)dx$$
$$=11.3889$$
$$\approx 11.4 \text{ sq units}$$

Question 13 **(14 marks)**

(a) Three points have coordinates $P(2, 1, 1)$, $Q(1, 0, -2)$ and $R(2, -1, 0)$.

(i) Determine the vector equation of the straight line through P and Q. (2 marks)

$$QP = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

(ii) Determine a unit vector that is perpendicular to the plane containing P, Q and R. (3 marks)

$$RP = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{n} = QP \times RP = \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}$$

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{30}} \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}$$

(iii) Show that the plane containing P, Q and R is parallel to the plane with Cartesian equation $10x + 2y - 4z = 1$. (3 marks)

Normal to plane through PQR is $\begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}$.

$$10x + 2y - 4z = 1 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \bullet \begin{bmatrix} 10 \\ 2 \\ -4 \end{bmatrix} = 1$$

Since $\begin{bmatrix} 10 \\ 2 \\ -4 \end{bmatrix} = -2 \times \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}$ then normals to both planes are parallel and so planes must be parallel.

(b) Consider the following system of linear equations, where k is a constant:

$$\begin{aligned} x+y+kz&=6 \\ -x+y+2kz&=0 \\ 2x+y+2kz&=6 \end{aligned}$$

(i) Clearly show that the system of equations has a unique solution provided $k \neq 0$.
(4 marks)

$$\begin{matrix} 1 & 1 & k & 6 \\ -1 & 1 & 2k & 0 \\ 2 & 1 & 2k & 6 \end{matrix}$$

$R_1 + R_2 \rightarrow R_2$
$2R_1 - R_3 \rightarrow R_3$

$$\begin{matrix} 1 & 1 & k & 6 \\ 0 & 2 & 3k & 6 \\ 0 & 1 & 0 & 6 \end{matrix}$$

$y = 6$

$12 + 3kz = 6 \Rightarrow z = \dfrac{-2}{k}$

$x + 6 + \dfrac{-2}{k} \times k = 6 \Rightarrow x = 2$

It can be seen that $x = 2$ and $y = 6$, but $z = \dfrac{-2}{k}$ and so $k \neq 0$ for z to be defined.

(ii) Describe briefly the geometric interpretation of the system of equations when $k = 0$.
(2 marks)

If $k = 0$ then system of equations reduces to three non-parallel straight lines in the x-y plane that intersect at three points.

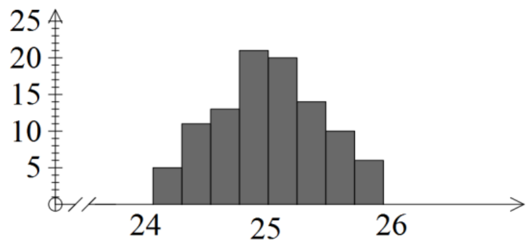
Question 14 **(8 marks)**

(a) A student carried out a simulation to select a random sample of 100 packets of nails and to record the mean length of the 40 nails in each packet. The nails were assumed to have a mean length of 25 mm and a standard deviation of 2.5 mm.

(i) Describe the expected features of a frequency graph showing the distribution of the 100 sample means. (3 marks)

- distribution of sample means would be normal
- distribution would be centred on $\bar{x} = 25$ mm
- distribution would have standard deviation of $\dfrac{2.5}{\sqrt{40}} \approx 0.395$ mm

(ii) Sketch a possible frequency graph with the features described in (i). (2 marks)



(Marks for scales and shape)

(b) Another student obtained a random sample of 100 journey times, in minutes, made by students on their way to school and calculated a 99% confidence interval for the mean journey time to be (17.46, 20.14). Determine the sample mean and estimate the standard deviation of the all journey times. (3 marks)

$$\bar{x} = \frac{20.14 + 17.46}{2} = 18.8 \text{ minutes}$$

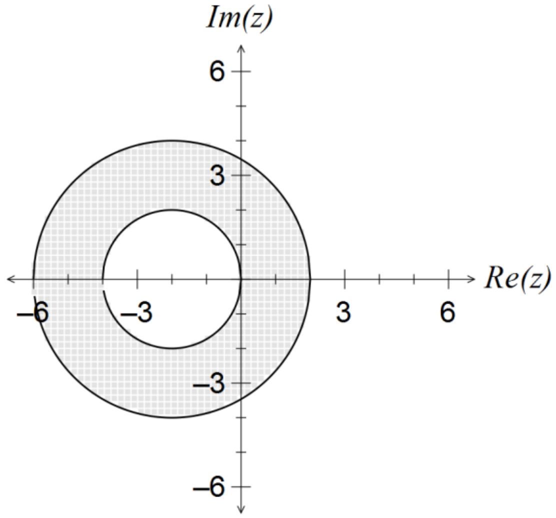
$$E = 18.8 - 17.46 = 1.34$$

$$2.576 \times \frac{s}{\sqrt{100}} = 1.34 \Rightarrow s = 5.2 \text{ minutes}$$

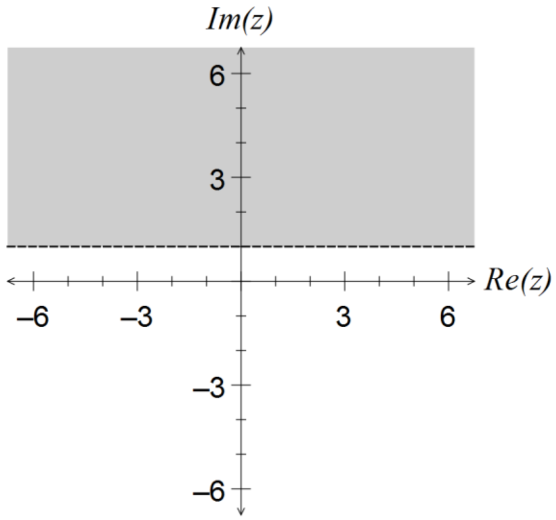
Question 15 **(8 marks)**

(a) Sketch the following regions in the complex plane.

(i) $2 \le |z+2| \le 4$ (3 marks)



(ii) $z - \bar{z} > 2i$ (3 marks)



(b) The set of points in the complex plane that satisfy $z - \bar{z} > 2i$ can also be described by $|z - 2 + i| > |z + a + bi|$, where a and b are real constants. State the values of a and b.

(2 marks)

$a = -2$

$b = -3$

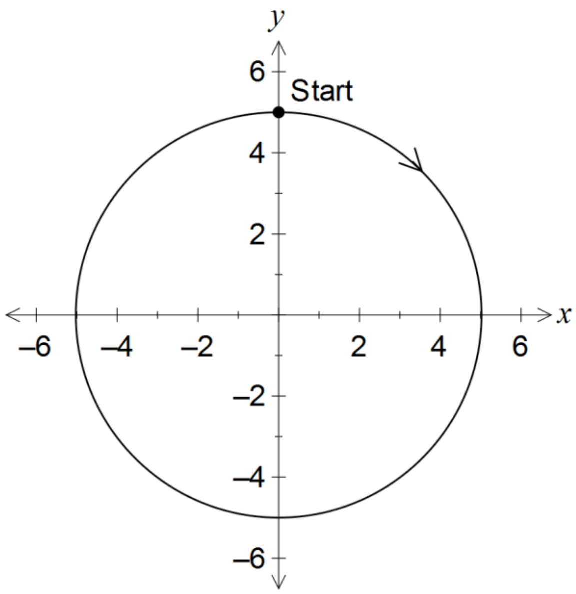
Question 16 **(7 marks)**

The position vector $\mathbf{r}$ of a particle at time t seconds, $t \geq 0$, is $\mathbf{r}(t) = 5\sin\left(\frac{\pi t}{3}\right)\mathbf{i} + 5\cos\left(\frac{\pi t}{3}\right)\mathbf{j}$.

(a) Determine the velocity of the particle at any time t. (1 mark)

$$\mathbf{v}(t) = \frac{5\pi}{3}\cos\left(\frac{\pi t}{3}\right)\mathbf{i} - \frac{5\pi}{3}\sin\left(\frac{\pi t}{3}\right)\mathbf{j}$$

(b) Sketch the path of the particle on the axes below, clearly indicating the initial position of the particle and the direction of motion. (3 marks)



(c) Show that the particle moves with a constant speed. (2 marks)

$$|\mathbf{v}| = \sqrt{\left(\frac{5\pi}{3}\cos\left(\frac{\pi t}{3}\right)\right)^2 + \left(-\frac{5\pi}{3}\sin\left(\frac{\pi t}{3}\right)\right)^2}$$

$$= \frac{5\pi}{3}\sqrt{\cos^2\left(\frac{\pi t}{3}\right) + \sin^2\left(\frac{\pi t}{3}\right)} = \frac{5\pi}{3}$$

(d) Determine the distance travelled by the particle in the first four seconds. (1 mark)

$$s = 4 \times \frac{5\pi}{3} = \frac{20\pi}{3}$$

Question 17 **(6 marks)**

(a) If $z = 2\cos\left(\frac{\pi}{3}\right) + 2i\sin\left(\frac{\pi}{3}\right)$, determine the reciprocal, $\frac{1}{z}$, in polar form. (2 marks)

$$z = 2cis\frac{\pi}{3}$$
$$z^{-1} = 2^{-1}cis$$
$$= \frac{1}{2}cis\left(-\frac{\pi}{3}\right)$$

(b) Let the non-zero complex number $z = r \cdot cis\theta$. Show that $\frac{1}{r \cdot cis\theta} = \frac{\overline{z}}{|z|^2}$. (2 marks)

$$\begin{aligned} LHS &= \frac{1}{r \cdot cis\theta} \\ &= \frac{1}{r \cdot cis\theta} \times \frac{r \cdot cis(-\theta)}{r \cdot cis(-\theta)} \\ &= \frac{r \cdot cis(-\theta)}{r^2} \\ &= \frac{\overline{z}}{|z|^2} \\ &= RHS \end{aligned}$$

(c) Describe the geometrical relationship between any non-zero complex number and its reciprocal. (2 marks)

The reciprocal, z^{-1}, will be the reflection of z in the real axis and then dilated about the origin by scale factor $\frac{1}{|z|^2}$.

Question 18 **(9 marks)**

The expected uptake of a new model of smartphone in a country, currently with one million models in use, can be modelled by the logistic equation $\frac{dx}{dt} = \frac{x(20-x)}{250}$, where x is the total number of models in millions and t is the time in weeks.

(a) Express x as a function of t in the form $x = \frac{a}{1 + b \cdot e^{-ct}}$, where a, b and c are positive constants. (5 marks)

$$\int \frac{dx}{x(20-x)} = \int \frac{dt}{250}$$

$$\int \frac{1}{x} + \frac{1}{20-x}\, dx = \int \frac{2}{25}\, dt$$

$$\ln|x| - \ln|20-x| = 0.08t + c$$

$$\frac{x}{20-x} = ae^{0.08t}$$

$$\frac{20-x}{x} = ke^{-0.08t},\ t = 0,\ x = 1 \ \Rightarrow\ k = 19$$

$$x = \frac{20}{1 + 19e^{-0.08t}}$$

(b) Calculate

(i) the expected number of models in use after 30 weeks. (1 mark)

$$\frac{20}{1 + 19e^{-0.08(30)}} = 7.343 \ \Rightarrow\ 7\ 343\ 000 \text{ models}$$

(ii) the week during which the number of models in use is increasing at the greatest rate. (3 marks)

For max $\frac{dx}{dt} \ \Rightarrow\ \frac{d^2x}{dt^2} = 0 \ \Rightarrow\ x = 10$

$$\frac{20}{1 + 19e^{-0.08t}} = 10 \ \Rightarrow\ t = 36.8$$

During 37th week.

See next page

Question 19 **(7 marks)**

A simple pendulum has a point mass on the end of a long string.

The mass, initially stationary, is released from position A. The mass swings a total distance of 48 cm away from A and then swings back again, returning to its initial position after a total of six seconds. The pendulum repeats this motion, with no measurable loss in amplitude from swing to swing.

The horizontal distance, x cm, of the point mass from A, t seconds after release, can be modelled by the equation $x = c - a\cos(nt)$, where a, n and c are positive constants.

(a) Express the acceleration $\frac{d^2x}{dt^2}$ in terms of x, n and c. (2 marks)

$$\frac{dx}{dt} = n \cdot a\sin(nt)$$
$$\frac{d^2x}{dt^2} = n^2 \cdot a\cos(nt)$$
$$= n^2(c - x)$$

(b) Determine the values of the positive constants a, n, and c. (3 marks)

Amplitude: $a = 48 \div 2 = 24$.

Period: $n = \frac{2\pi}{6} = \frac{\pi}{3}$.

Shift: $c = 24$.

(c) At what time(s) during the first ten seconds is the mass moving with maximum speed? (2 marks)

From (a), $c - x = 0$ when $x = 24$.

Solve $24 - 24\cos\left(\frac{\pi t}{3}\right) = 24$.

When $t = 1.5,\ 4.5$ and 7.5 seconds.

Question 20 **(7 marks)**

At 0830 on a windless day, a light aircraft takes off from an airport and flies due east at a constant 300 km/h. At the same instant, a passenger aircraft is 50 km due south of the airport and is flying directly towards the airport at a constant speed of 400 km/h.

Determine the rate, to the nearest km/h, that the distance between the two aircraft is changing at the instant the passenger plane is 30 km from the airport.

Let a and b be distance of light and passenger aircraft from airport respectively.

$$a = 300t$$

$$b = 50 - 400t$$

$$30 = 50 - 400t \Rightarrow t = \frac{1}{20} \text{ h}$$

$$x^2 = a^2 + b^2$$
$$= (300t)^2 + (50 - 400t)^2$$

$$x = 50\sqrt{100t^2 - 16t + 1}$$

$$\frac{dx}{dt} = \frac{25(200t - 16)}{\sqrt{100t^2 - 16t + 1}}$$

$$t = \frac{1}{20}$$
$$\frac{dx}{dt} = -100\sqrt{5} \text{ km/h}$$
$$\approx -223.6 \text{ km/h}$$

Distance is decreasing at 224 km/h.

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________